

A NOTE ON THE BASIC LICHNEROWICZ COHOMOLOGY OF TRANSVERSALLY KÄHLERIAN FOLIATIONS

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ABSTRACT. In this paper we study basic Lichnerowicz cohomology of Kählerian foliations. An $(0, 2)$ -basic Euler class and an associated Gysin sequence of Kählerian flows are investigated. Also, a basic Lichnerowicz cohomology of transversally locally conformally Kählerian foliations is studied as well as its relation with basic Bott-Chern cohomology and basic holomorphic cohomology of a foliated complex line bundle.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. The importance of the Lichnerowicz cohomology, also known as Morse-Novikov cohomology (defined independently by Lichnerowicz in [17] and Novikov in [20]), is without question and it was intensively studied by many authors, e.g. [2, 17, 20, 24, 33].

The aim of the present paper is to extend at basic forms of Kählerian foliations $(\mathcal{M}, \mathcal{F})$ some problems related to Lichnerowicz cohomology theory. In this sense, in the first section, following [5, 6], we make a short review on the de Rham and Dolbeault theory for basic forms of Kählerian foliations. The second section is dedicated to study of the basic Lichnerowicz cohomology of Kählerian foliations. Following [33], we present a basic version of a de Rham theorem for basic Lichnerowicz cohomology, we discuss about an associated twisted basic cohomology and we consider the Dolbeault and Bott-Chern cohomology of the basic Lichnerowicz complex. Also a relative basic Lichnerowicz cohomology is studied. Next, following Prieto Royo [25], we define the $(0, 2)$ -basic Euler class of a Kählerian flow and we construct an associated Gysin sequence. Also a Hodge decomposition for Dolbeault-Lichnerowicz cohomology associated to a Kählerian flow is given. Finally, we study the basic Lichnerowicz cohomology of transversally locally conformally Kählerian foliations, we present three basic cohomological invariants of such structures and the relation with the basic Bott-Chern cohomology and basic holomorphic cohomology of a foliated complex line bundle is discussed, giving a basic version of some results of Ornea and Verbitsky [22] from the case of locally conformally Kähler manifolds.

1.2. Preliminaries. Let us consider \mathcal{M} an $(n + m)$ -dimensional manifold which will be assumed to be connected and orientable. Differential forms (and in particular functions) will take their values in the field of complex numbers \mathbb{C} . If φ is a form, then $\overline{\varphi}$ denote its complex conjugate and we say that φ is *real* if $\varphi = \overline{\varphi}$.

Definition 1.1. A codimension n foliation \mathcal{F} on \mathcal{M} is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

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- (i) $\{U_i\}$, $i \in I$ is an open covering of \mathcal{M} ;
- (ii) For every $i \in I$, $\varphi_i : U_i \rightarrow M$ are submersions, where M is an n -dimensional manifold, called transversal manifold;
- (iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$ satisfy

$$(1.1) \quad \varphi_j = f_{i,j} \circ \varphi_i$$

for every $(i, j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$.

Every fibre of φ_i is called a *plaque* of the foliation. Condition (1.1) says that, on the intersection $U_i \cap U_j$ the plaques defined respectively by φ_i and φ_j are the "same". The manifold M is decomposed into a family of disjoint immersed connected submanifolds of dimension m ; each of these submanifolds is called a *leaf* of \mathcal{F} .

We say that \mathcal{F} is *transversally orientable* if on \mathcal{M} can be given an orientation which is preserved by all $f_{i,j}$. By $T\mathcal{F}$ we denote the tangent bundle to \mathcal{F} and $\Gamma(\mathcal{F})$ is the space of its global sections i.e. vector fields tangent to \mathcal{F} . We say that a differential form φ is *basic* if it satisfies $i_X \varphi = \mathcal{L}_X \varphi = 0$ for every $X \in \Gamma(\mathcal{F})$, where i_X and \mathcal{L}_X denotes the the interior product and Lie derivative with respect to X , respectively. A *basic function* is a function constant on the leaves; such functions form an algebra denoted by $\mathcal{F}_b(\mathcal{M})$. The quotient $Q\mathcal{F} = T\mathcal{M}/T\mathcal{F}$ is the normal bundle of \mathcal{F} . A vector field $Y \in \mathcal{X}(\mathcal{M})$ is said to be *foliated* if, for every $X \in \Gamma(\mathcal{F})$ we have $[X, Y] \in \Gamma(\mathcal{F})$; $\mathcal{X}(\mathcal{M}, \mathcal{F})$ denotes the algebra of foliated vector fields on \mathcal{M} . The quotient $\mathcal{X}(\mathcal{M}/\mathcal{F}) = \mathcal{X}(\mathcal{M}, \mathcal{F})/\Gamma(\mathcal{F})$ is called the algebra of *basic vector fields* on \mathcal{M} .

In this paper a system of local coordinates adapted to the foliation \mathcal{F} means coordinates $(z^1, \dots, z^n, y^1, \dots, y^m)$ on an open subset U on which the foliation is trivial and defined by the equations $dz^i = 0$, $i = 1, \dots, n$. If \mathcal{F} is transversally holomorphic (see definition (1.2.2) below) z^1, \dots, z^n will be complex coordinates.

Definition 1.2. A transverse structure to \mathcal{F} is a geometric structure on M invariant by all the local diffeomorphisms $f_{i,j}$.

A transverse structure can be considered as a geometric structure on the leaf space \mathcal{M}/\mathcal{F} (which is not a manifold in general).

- 2.2.1. If M is a Riemannian manifold and all the $f_{i,j}$ are isometries then \mathcal{F} is said to be Riemannian. This means that the normal bundle $Q\mathcal{F}$ is equipped with a Riemannian metric which is "invariant along the leaves".
- 2.2.2. If M is a complex manifold and all the $f_{i,j}$ are biholomorphic maps then we say that \mathcal{F} is transversally holomorphic. In that case, any transversal to \mathcal{F} inherits a complex structure.
- 2.2.3. If M is a Hermitian manifold and all the $f_{i,j}$ preserve the Hermitian structure then we say that \mathcal{F} is Hermitian. (The $f_{i,j}$ are in particular biholomorphic maps and isometries.) The normal bundle $Q\mathcal{F}$ is equipped with a Hermitian metric "invariant along the leaves".
- 2.2.4. If M is a Kählerian manifold and all the $f_{i,j}$ preserve the Kähler structure we say that \mathcal{F} is transversally Kählerian. In particular such a foliation is Hermitian. This is equivalent to the existence of a Hermitian metric g on the normal bundle $Q\mathcal{F}$ which can be written in a transverse local system of coordinates (z^1, \dots, z^n) in the form $g = g_{j\bar{k}}(z, \bar{z}) dz^j \otimes d\bar{z}^k$ such that its skew-symmetric part $\omega = \frac{i}{2} g_{j\bar{k}}(z, \bar{z}) dz^j \otimes d\bar{z}^k$ is closed (ω is a basic 2-form called the basic Kähler form of \mathcal{F}).

Throughout this paper we consider \mathcal{F} to be transversally holomorphic with $2n$ codimension. Let $\Omega^r(\mathcal{M}/\mathcal{F})$ be the space of all basic forms of degree r . It is easy to see that the exterior derivative of a basic form is also a basic form. Indeed, if $\varphi \in \Omega^r(\mathcal{M}/\mathcal{F})$ then $i_X \varphi = \mathcal{L}_X \varphi = 0$ for any $X \in \Gamma(\mathcal{F})$ and, then from remarkable identity $\mathcal{L}_X = i_X d + di_X$ and $d^2 = 0$ it follows that $i_X d\varphi = \mathcal{L}_X d\varphi = 0$ for any $X \in \Gamma(\mathcal{F})$. Let us denote by $d_b = d|_{\Omega^\bullet(\mathcal{M}/\mathcal{F})}$ the restriction of exterior derivative to basic forms. Then we have $d_b : \Omega^\bullet(\mathcal{M}/\mathcal{F}) \rightarrow \Omega^{\bullet+1}(\mathcal{M}/\mathcal{F})$ and the differential complex

$$(1.2) \quad 0 \rightarrow \Omega^0(\mathcal{M}/\mathcal{F}) \xrightarrow{d_b} \Omega^1(\mathcal{M}/\mathcal{F}) \xrightarrow{d_b} \dots \xrightarrow{d_b} \Omega^{2n}(\mathcal{M}/\mathcal{F}) \rightarrow 0$$

which is called the *basic de Rham complex* of \mathcal{F} ; its cohomology is the basic de Rham cohomology $H^\bullet(\mathcal{M}/\mathcal{F})$. Now, we consider $Q_{\mathbb{C}}\mathcal{F} = Q\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified normal bundle of \mathcal{F} . Let J be the automorphism of $Q_{\mathbb{C}}\mathcal{F}$ associated to the complex structure; J satisfies $J^2 = -\text{Id}$ and then has two eigenvalues i and $-i$ with associated eigensubbundles respectively denoted by $Q^{1,0}\mathcal{F}$ and $Q^{0,1}\mathcal{F} = \overline{Q^{1,0}\mathcal{F}}$. We have a splitting $Q_{\mathbb{C}}\mathcal{F} = Q^{1,0}\mathcal{F} \oplus Q^{0,1}\mathcal{F}$ which gives rise to decomposition

$$\Lambda^r(Q_{\mathbb{C}}^*\mathcal{F}) = \bigoplus_{p+q=r} \Lambda^{p,q},$$

where $\Lambda^{p,q} = \Lambda^p(Q^{1,0*}\mathcal{F}) \otimes \Lambda^q(Q^{0,1*}\mathcal{F})$. Basic sections of $\Lambda^{p,q}$ are called *basic forms of type (p, q)* . They form a vector space denoted by $\Omega^{p,q}(\mathcal{M}/\mathcal{F})$. We have

$$(1.3) \quad \Omega^r(\mathcal{M}/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(\mathcal{M}/\mathcal{F}).$$

As in the classical case of a complex manifold, see [19], the basic exterior derivative decomposes into a sum of two operators

$$\partial_b : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \rightarrow \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}), \quad \overline{\partial}_b : \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \rightarrow \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}).$$

We have $\partial_b^2 = \overline{\partial}_b^2 = 0$ and $\partial_b \overline{\partial}_b + \overline{\partial}_b \partial_b = 0$. The differential complex

$$(1.4) \quad 0 \rightarrow \Omega^{p,0}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \Omega^{p,1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \dots \xrightarrow{\overline{\partial}_b} \Omega^{p,n}(\mathcal{M}/\mathcal{F}) \rightarrow 0$$

is called the *basic Dolbeault complex* of \mathcal{F} ; its cohomology $H^{p,\bullet}(\mathcal{M}/\mathcal{F})$ is the basic Dolbeault cohomology of foliation \mathcal{F} .

2. BASIC LICHNEROWICZ COHOMOLOGY OF TRANSVERSALLY KÄHLERIAN FOLIATIONS

2.1. Basic Lichnerowicz cohomology. Let $(\mathcal{M}, \mathcal{F})$ be a transversally holomorphic foliation and $\theta \in \Omega^1(\mathcal{M}/\mathcal{F})$ be a d_b -closed basic 1-form. Denote by $d_{b,\theta} : \Omega^r(\mathcal{M}/\mathcal{F}) \rightarrow \Omega^{r+1}(\mathcal{M}/\mathcal{F})$ the map $d_{b,\theta} = d_b - \theta \wedge$.

Since $d_b \theta = 0$, we easily obtain that $d_{b,\theta}^2 = 0$. The differential complex

$$(2.1) \quad 0 \rightarrow \Omega^0(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \Omega^1(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \dots \xrightarrow{d_{b,\theta}} \Omega^{2n}(\mathcal{M}/\mathcal{F}) \rightarrow 0$$

is called the *basic Lichnerowicz complex* of $(\mathcal{M}, \mathcal{F})$; its cohomology groups $H_\theta^\bullet(\mathcal{M}/\mathcal{F}) := H_L^\bullet(\mathcal{M}/\mathcal{F})$ are called the *basic Lichnerowicz cohomology groups* of $(\mathcal{M}, \mathcal{F})$.

This is a basic version of the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [17] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic geometry manifolds (see [2, 16]). We also notice that Vaisman in [33] studied it under the name of "adapted cohomology" on locally conformal Kähler (LCK) manifolds.

Some notions concerning to a such basic Lichnerowicz cohomology of real foliations may be found in [11].

We notice that, locally, the basic Lichnerowicz cohomology complex becomes the basic de Rham complex after a change $\varphi \mapsto e^f \varphi$ with f a basic function which satisfies $d_b f = \theta$, namely $d_{b,\theta}$ is the unique differential in $\Omega^\bullet(\mathcal{M}/\mathcal{F})$ which makes the multiplication by the smooth basic function e^f an isomorphism of cochain basic complexes $e^f : (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta}) \rightarrow (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_b)$.

Proposition 2.1. *The basic Lichnerowicz cohomology depends only on the basic class of θ . In fact, we have the following isomorphism $H_{\theta-d_b f}^r(\mathcal{M}/\mathcal{F}) \approx H_\theta^r(\mathcal{M}/\mathcal{F})$.*

Proof. Since $d_{b,\theta}(e^f \varphi) = e^f d_{b,\theta-d_b f} \varphi$ it results that the map $[\varphi] \mapsto [e^f \varphi]$ is an isomorphism between $H_{\theta-d_b f}^r(\mathcal{M}/\mathcal{F})$ and $H_\theta^r(\mathcal{M}/\mathcal{F})$. \square

Using the definition of $d_{b,\theta}$ we easily obtain

$$d_{b,\theta}(\varphi \wedge \psi) = d_b \varphi \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge d_{b,\theta} \psi.$$

Also, if θ_1 and θ_2 are two d_b -closed basic 1-forms then

$$d_{b,\theta_1+\theta_2}(\varphi \wedge \psi) = d_{b,\theta_1} \varphi \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge d_{b,\theta_2} \psi,$$

which says that the wedge product induces the map

$$\wedge : H_{\theta_1}^{r_1}(\mathcal{M}/\mathcal{F}) \times H_{\theta_2}^{r_2}(\mathcal{M}/\mathcal{F}) \rightarrow H_{\theta_1+\theta_2}^{r_1+r_2}(\mathcal{M}/\mathcal{F}).$$

Corollary 2.1. *The wedge product induces the following homomorphism*

$$\wedge : H_\theta^r(\mathcal{M}/\mathcal{F}) \times H_{-\theta}^r(\mathcal{M}/\mathcal{F}) \rightarrow H^{2r}(\mathcal{M}/\mathcal{F}).$$

According to [11], we have

Proposition 2.2. *If $\theta \in \Omega^1(\mathcal{M}/\mathcal{F})$ is d_b -closed and no d_b -exact, then $H_\theta^0(\mathcal{M}/\mathcal{F}) = 0$.*

Proposition 2.3. *The inclusion $i : \Omega^\bullet(\mathcal{M}/\mathcal{F}) \rightarrow \Omega^\bullet(\mathcal{M})$ induce an injective map in cohomology $H_\theta^1(\mathcal{M}/\mathcal{F}) \rightarrow H_\theta^1(\mathcal{M})$.*

Now, if we restrict some considerations from [33] at basic forms it results that the basic cohomology spaces $H_\theta^r(\mathcal{M}/\mathcal{F})$ can also be obtained as the basic cohomology spaces of $(\mathcal{M}, \mathcal{F})$ with the coefficients in the sheaf of germs of smooth $d_{b,\theta}$ -closed basic functions. Namely, let us denote by $\mathcal{F}_{b,\theta}(\mathcal{M})$ the sheaf of germs of smooth basic functions on $(\mathcal{M}, \mathcal{F})$ which are such $d_{b,\theta} f = d_b f - f \theta = 0$.

Firstly, we notice that $d_{b,\theta}$ satisfies a Poincaré type Lemma for basic forms. Then if we denote by $\mathcal{A}^r(\mathcal{M}/\mathcal{F})$ the sheaf of germs of differentiable basic r -forms on $(\mathcal{M}, \mathcal{F})$, we see that

$$(2.2) \quad 0 \longrightarrow \mathcal{F}_{b,\theta}(\mathcal{M}) \xrightarrow{i} \mathcal{A}^0(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \mathcal{A}^1(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \dots$$

is a fine resolution of $\mathcal{F}_{b,\theta}(\mathcal{M})$, which leads to

Proposition 2.4. *(A basic de Rham type Theorem). For every transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ and every d_b -closed basic 1-form θ , one has the isomorphisms*

$$H^r(\mathcal{M}/\mathcal{F}, \mathcal{F}_{b,\theta}(\mathcal{M})) \approx H_\theta^r(\mathcal{M}/\mathcal{F}).$$

For every θ as above, let us consider now the auxiliary basic operator $\tilde{d}_b = d_b - \frac{r}{2}\theta \wedge$ where r is the degree of the basic form acted on. We notice that \tilde{d}_b is an antiderivation of basic differential forms and it is easy to see that $\tilde{d}_b^2 = -\frac{1}{2}\theta \wedge d_b$. Then \tilde{d}_b defines a *twisted basic cohomology*, [34], of basic differential forms of $(\mathcal{M}, \mathcal{F})$, which is given by

$$(2.3) \quad H_{\tilde{d}_b}^\bullet(\mathcal{M}/\mathcal{F}) = \frac{\text{Ker } \tilde{d}_b}{\text{Im } \tilde{d}_b \cap \text{Ker } \tilde{d}_b}$$

and is isomorphic to the cohomology of the cochain basic complex $(\tilde{\Omega}^\bullet(\mathcal{M}/\mathcal{F}), \tilde{d}_b)$ consisting of the basic differential forms $\varphi \in \Omega^\bullet(\mathcal{M}/\mathcal{F})$ satisfying $\tilde{d}_b^2 \varphi = -\theta \wedge d_b \varphi = 0$.

The basic complex $\tilde{\Omega}^\bullet(\mathcal{M}/\mathcal{F})$ admits a basic subcomplex $\Omega_\theta^\bullet(\mathcal{M}/\mathcal{F})$, namely, the ideal generated by θ . On this subcomplex, $\tilde{d}_b = d_b$, which means that it is a basic subcomplex of the usual basic de Rham complex of $(\mathcal{M}, \mathcal{F})$. Hence, one has the homomorphisms

$$(2.4) \quad a : H^r(\Omega_\theta^\bullet(\mathcal{M}/\mathcal{F})) \rightarrow H_{\tilde{d}_b}^r(\mathcal{M}/\mathcal{F}), \quad b : H^r(\Omega_\theta^\bullet(\mathcal{M}/\mathcal{F})) \rightarrow H^r(\mathcal{M}/\mathcal{F}, \mathbb{R}).$$

Now, we can easily construct a homomorphism

$$(2.5) \quad c : H_{\tilde{d}_b}^r(\mathcal{M}/\mathcal{F}) \rightarrow H^{r+1}(\mathcal{M}/\mathcal{F}, \mathbb{R}).$$

Indeed, if $[\varphi] \in H_{\tilde{d}_b}^r(\mathcal{M}/\mathcal{F})$, where φ is \tilde{d}_b -closed basic form, then we put $c([\varphi]) = [\theta \wedge \varphi]$, and this produces the homomorphism from (2.5). We notice that the existence of c gives some relation between \tilde{d}_b and the basic real cohomology of $(\mathcal{M}, \mathcal{F})$.

For the basic Lichnerowicz cohomology, similar basic complexes of Dolbeault and Bott-Chern type can be defined. Taking into account the decomposition $\theta = \theta^{1,0} + \theta^{0,1}$, consider the Hodge components of the basic Lichnerowicz differential $d_{b,\theta} = d_b - \theta \wedge$ as

$$(2.6) \quad d_{b,\theta} = \partial_{b,\theta} + \bar{\partial}_{b,\theta}, \quad \partial_{b,\theta} = \partial_b - \theta^{1,0} \wedge, \quad \bar{\partial}_{b,\theta} = \bar{\partial}_b - \theta^{0,1} \wedge.$$

The differential complex

$$(2.7) \quad \dots \xrightarrow{\bar{\partial}_{b,\theta}} \Omega^{p,q-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\bar{\partial}_{b,\theta}} \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \xrightarrow{\bar{\partial}_{b,\theta}} \dots$$

is called *the basic Dolbeault-Lichnerowicz complex* of $(\mathcal{M}, \mathcal{F})$; its cohomology groups denoted by $H_\theta^{p,\bullet}(\mathcal{M}/\mathcal{F}) := H_{DL}^{p,\bullet}(\mathcal{M}/\mathcal{F})$ are called *the basic Dolbeault-Lichnerowicz cohomology groups* of $(\mathcal{M}, \mathcal{F})$.

The differential complex

$$(2.8) \quad \dots \Omega^{p-1,q-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\partial_{b,\theta} \oplus \bar{\partial}_{b,\theta}} \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \xrightarrow{\partial_{b,\theta} \oplus \bar{\partial}_{b,\theta}} \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}) \oplus \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}) \dots$$

is called *the basic Bott-Chern-Lichnerowicz complex* of $(\mathcal{M}, \mathcal{F})$ and its cohomology groups

$$H_{BCL}^{\bullet,\bullet}(\mathcal{M}/\mathcal{F}) = \frac{\text{Ker}\{\Omega^{\bullet,\bullet} \xrightarrow{\partial_{b,\theta}} \Omega^{\bullet+1,\bullet}\} \cap \text{Ker}\{\Omega^{\bullet,\bullet} \xrightarrow{\bar{\partial}_{b,\theta}} \Omega^{\bullet,\bullet+1}\}}{\text{Im}\{\Omega^{\bullet-1,\bullet-1} \xrightarrow{\partial_{b,\theta} \oplus \bar{\partial}_{b,\theta}} \Omega^{\bullet,\bullet}\}}$$

are called *the basic Bott-Chern-Lichnerowicz cohomology groups* of $(\mathcal{M}, \mathcal{F})$.

Similar as above, one gets

Proposition 2.5. *The basic Dolbeault-Lichnerowicz cohomology depends only on the basic Dolbeault class of $\theta^{0,1}$. In fact, we have the following isomorphism $H_{\theta-d_b f}^{p,q}(\mathcal{M}/\mathcal{F}) \approx H_\theta^{p,q}(\mathcal{M}/\mathcal{F})$.*

Corollary 2.2. *The wedge product induces the following homomorphism*

$$\wedge : H_{\theta}^{p,q}(\mathcal{M}/\mathcal{F}) \times H_{-\theta}^{p,q}(\mathcal{M}/\mathcal{F}) \rightarrow H^{2p,2q}(\mathcal{M}/\mathcal{F}).$$

Proposition 2.6. *If $\theta \in \Omega^1(\mathcal{M}/\mathcal{F})$ is d_b -closed and no d_b -exact, then $H_{\theta}^{0,0}(\mathcal{M}/\mathcal{F}) = 0$.*

Proposition 2.7. *The inclusion $i : \Omega^{\bullet,\bullet}(\mathcal{M}/\mathcal{F}) \rightarrow \Omega^{\bullet,\bullet}(\mathcal{M})$ induce an injective map in cohomology $H_{\theta}^{0,1}(\mathcal{M}/\mathcal{F}) \rightarrow H_{\theta}^{0,1}(\mathcal{M})$.*

Proposition 2.8. *(A basic Dolbeault type Theorem). For every transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ and every d_b -closed basic 1-form θ , one has the isomorphisms*

$$H^{p,q}(\mathcal{M}/\mathcal{F}, \mathcal{F}_{b,\theta^{0,1}}(\mathcal{M})) \approx H_{\theta}^{p,q}(\mathcal{M}/\mathcal{F}),$$

where $\mathcal{F}_{b,\theta^{0,1}}(\mathcal{M})$ is the sheaf of germs of smooth basic functions on $(\mathcal{M}, \mathcal{F})$ which satisfy $\bar{\partial}_{b,\theta} f = \bar{\partial}_b f - f\theta^{0,1} = 0$.

We also notice that if we consider the decomposition $\tilde{d}_b = \tilde{\partial}_b + \tilde{\bar{\partial}}_b$ we can construct analogous of homomorphisms a, b and c from (2.4) and (2.5), respectively, for corresponding basic Dolbeault cohomology.

2.2. Relative basic Lichnerowicz cohomology. The relative de Rham cohomology was defined in [4] p. 78. In this subsection we construct a similar version for our basic Lichnerowicz cohomology.

Let $(\mathcal{M}, \mathcal{F})$ and $(\mathcal{M}', \mathcal{F}')$ be two foliated manifolds.

Definition 2.1. A *morphism* from $(\mathcal{M}, \mathcal{F})$ to $(\mathcal{M}', \mathcal{F}')$ is a differentiable mapping $\mu : \mathcal{M} \rightarrow \mathcal{M}'$ which sends every leaf F of \mathcal{F} into a leaf F' of \mathcal{F}' such that the restriction map $\mu : F \rightarrow F'$ is smooth, and it is holomorphic in transverse coordinates if \mathcal{F} and \mathcal{F}' are transversally holomorphic.

We also notice that μ^* preserves the basic forms.

By restricted of the standard relation $d\mu^* = \mu^*d'$ to basic forms, (here d' denotes the exterior derivative on \mathcal{M}'), we obtain

$$(2.9) \quad d_b \mu^* = \mu^* d'_b.$$

Now, if θ is a d'_b -closed basic 1-form on $(\mathcal{M}', \mathcal{F}')$, using (2.9), a straightforward calculus leads to a similar desired relation, namely

$$(2.10) \quad d_{b,\mu^*\theta} \mu^* = \mu^* d'_{b,\theta}.$$

Indeed, for $\varphi \in \Omega^r(\mathcal{M}'/\mathcal{F}')$, we have

$$\begin{aligned} d_{b,\mu^*\theta}(\mu^*\varphi) &= d_b(\mu^*\varphi) - \mu^*\theta \wedge \mu^*\varphi \\ &= \mu^*(d'_b\varphi) - \mu^*(\theta \wedge \varphi) \\ &= \mu^*(d'_{b,\theta}\varphi). \end{aligned}$$

We define the differential complex $(\Omega^r(\mu), \tilde{d}_{b,\theta})$, where

$$\Omega^r(\mu) = \Omega^r(\mathcal{M}'/\mathcal{F}') \oplus \Omega^{r-1}(\mathcal{M}/\mathcal{F}), \text{ and } \tilde{d}_{b,\theta}(\varphi, \psi) = (d'_{b,\theta}\varphi, \mu^*\varphi - d_{b,\mu^*\theta}\psi).$$

Taking into account $d_{b,\theta}^2 = d_{b,\mu^*\theta}^2 = 0$ and (2.10) we easily verify that $\tilde{d}_{b,\theta}^2 = 0$. Denote the cohomology groups of this complex by $H_{\theta}^{\bullet}(\mu)$.

If we regraduate the complex $\Omega^r(\mathcal{M}/\mathcal{F})$ as $\tilde{\Omega}^r(\mathcal{M}/\mathcal{F}) = \Omega^{r-1}(\mathcal{M}/\mathcal{F})$, then we obtain an exact sequence of differential complexes

$$(2.11) \quad 0 \longrightarrow \tilde{\Omega}^r(\mathcal{M}/\mathcal{F}) \xrightarrow{\alpha} \Omega^r(\mu) \xrightarrow{\beta} \Omega^r(\mathcal{M}'/\mathcal{F}') \longrightarrow 0$$

with the obvious mappings α and β given by $\alpha(\psi) = (0, \psi)$ and $\beta(\varphi, \psi) = \varphi$, respectively. From (2.11) we have an exact sequence in cohomologies

$$\dots \longrightarrow H_{\mu^*\theta}^{r-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\alpha^*} H_{\theta}^r(\mu) \xrightarrow{\beta^*} H_{\theta}^r(\mathcal{M}'/\mathcal{F}') \xrightarrow{\delta^*} H_{\mu^*\theta}^r(\mathcal{M}/\mathcal{F}) \longrightarrow \dots$$

It is easily seen that $\delta^* = \mu^*$. Here μ^* denotes the corresponding map between cohomology groups. Let $\varphi \in \Omega^r(\mathcal{M}'/\mathcal{F}')$ be a $d'_{b,\theta}$ -closed form, and $(\varphi, \psi) \in \Omega^r(\mu)$. Then $\tilde{d}_{b,\theta}(\varphi, \psi) = (0, \mu^*\varphi - d_{b,\mu^*\theta}\psi)$ and by the definition of the operator δ^* we have

$$\delta^*[\varphi] = [\mu^*\varphi - d_{b,\mu^*\theta}\psi] = [\mu^*\varphi] = \mu^*[\varphi].$$

Hence we finally get a long exact sequence

$$(2.12) \quad \dots \longrightarrow H_{\mu^*\theta}^{r-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\alpha^*} H_{\theta}^r(\mu) \xrightarrow{\beta^*} H_{\theta}^r(\mathcal{M}'/\mathcal{F}') \xrightarrow{\mu^*} H_{\mu^*\theta}^r(\mathcal{M}/\mathcal{F}) \longrightarrow \dots$$

Thus, similar arguments as in [28], leads to

Proposition 2.9. *If the foliated manifolds $(\mathcal{M}, \mathcal{F})$ and $(\mathcal{M}', \mathcal{F}')$ are of the n -th and n' -th codimension, respectively, then*

- (i) $\beta^* : H_{\theta}^{n+1}(\mu) \rightarrow H_{\theta}^{n+1}(\mathcal{M}'/\mathcal{F}')$ is an epimorphism,
- (ii) $\alpha^* : H_{\mu^*\theta}^{n'}(\mathcal{M}/\mathcal{F}) \rightarrow H_{\theta}^{n'+1}(\mu)$ is an epimorphism,
- (iii) $\beta^* : H_{\theta}^r(\mu) \rightarrow H_{\theta}^r(\mathcal{M}'/\mathcal{F}')$ is an isomorphism for $r > n+1$,
- (iv) $\alpha^* : H_{\mu^*\theta}^r(\mathcal{M}/\mathcal{F}) \rightarrow H_{\theta}^{r+1}(\mu)$ is an isomorphism for $r > n'$,
- (v) $H_{\theta}^r(\mu) = 0$ for $r > \max\{n+1, n'\}$.

If \mathcal{F} and \mathcal{F}' are transversally holomorphic, taking into account that $d_{b,\theta} = \partial_{b,\theta} + \bar{\partial}_{b,\theta}$, from (2.10), one gets $\bar{\partial}_{b,\mu^*\theta}\mu^* = \mu^*\bar{\partial}'_{b,\theta}$. Then, in similar manner as above we can obtain corresponding results concerning to relative cohomology for basic Dolbeault-Lichnerowicz cohomology groups.

2.3. The basic Euler $(0, 2)$ class and Gysin sequence associated to a Kählerian flow. Recall that a flow on \mathcal{M} is a 1-dimensional oriented foliation \mathcal{F} on \mathcal{M} . A flow \mathcal{F} is *Riemannian* if there exists a holonomy invariant Riemannian metric G . Such a metric is said to be *bundle-like*. We can choose a smooth nonsingular vector field ξ defining \mathcal{F} such that $G(\xi, \xi) = 1$. We call the 1-form $\eta = i_{\xi}G$ the *characteristic form* of the flow. The *mean curvature* of \mathcal{F} is defined by $k = \mathcal{L}_{\xi}\eta$. These two forms depend on the flow \mathcal{F} and the metric G . It has been shown in [8] that we can choose a bundle-like metric G such that k is basic. In these conditions, k is closed, and defines a class $[k] \in H^1(\mathcal{M}/\mathcal{F})$, which does not depend on the metric G , see [1], and that vanishes if and only if the flow is isometric. This invariant is called *the Alvarez class* of \mathcal{F} . We also notice that we have the decomposition

$$(2.13) \quad d\eta = \mathcal{E}(\mathcal{F}) + \eta \wedge k,$$

and the 2-form $\mathcal{E}(\mathcal{F})$ thus defined is called *the basic Euler form* of \mathcal{F} , and satisfies

$$(2.14) \quad d_b\mathcal{E}(\mathcal{F}) + k \wedge \mathcal{E}(\mathcal{F}) = 0.$$

Thus $\mathcal{E}(\mathcal{F})$ defines an element $[\mathcal{E}(\mathcal{F})] \in H_{-k}^2(\mathcal{M}/\mathcal{F})$ called *the basic Euler class* of \mathcal{F} .

In the following, we consider that $(\mathcal{M}, \mathcal{F})$ is a transversally Kählerian flow endowed with a bundle-like metric G , $\dim_{\mathbb{R}} \mathcal{M} = 2n + 1$.

Because the all forms from (2.14) are basic, taking the $(0, 3)$ components, we get

$$(2.15) \quad \bar{\partial}_b \mathcal{E}^{0,2}(\mathcal{F}) + k^{0,1} \wedge \mathcal{E}^{0,2}(\mathcal{F}) = 0.$$

and the $(0, 2)$ -form $\mathcal{E}^{0,2}(\mathcal{F})$ is called *the basic Euler $(0, 2)$ -form* of \mathcal{F} . Thus $\mathcal{E}^{0,2}(\mathcal{F})$ defines an element $[\mathcal{E}^{0,2}(\mathcal{F})] \in H_{-k}^{0,2}(\mathcal{M}/\mathcal{F})$ called *the basic Euler $(0, 2)$ class* of \mathcal{F} . Here $k^{0,1}$ is the $(0, 1)$ -component of k .

The short Gysin sequence associated to the basic Dolbeault cohomology is the following sequence of differential complexes:

$$(2.16) \quad 0 \longrightarrow \Omega^{\bullet,\bullet}(\mathcal{M}/\mathcal{F}) \xrightarrow{i} \Omega^{\bullet,\bullet}(\mathcal{M}) \xrightarrow{\rho} \Omega^{\bullet,\bullet}(\mathcal{M})/\Omega^{\bullet,\bullet}(\mathcal{M}/\mathcal{F}) \longrightarrow 0$$

where ρ is the projection induced by the inclusion i . Then, similar arguments as in [25] leads to

Theorem 2.1. *Let \mathcal{F} be a Kählerian flow on the closed manifold \mathcal{M} , and choose a metric G with basic mean curvature form. Then, we have the following long exact sequence:*

$$(2.17) \quad \dots H^{p,q}(\mathcal{M}/\mathcal{F}) \longrightarrow H^{p,q}(\mathcal{M}) \longrightarrow H_k^{p,q-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{[\mathcal{E}^{0,2}(\mathcal{F})]} H^{p,q+1}(\mathcal{M}/\mathcal{F}) \dots,$$

where the connecting morphism is, up to sign, multiplication by the basic Euler $(0, 2)$ class $[\mathcal{E}^{0,2}(\mathcal{F})] \in H_{-k}^{0,2}(\mathcal{M}/\mathcal{F})$.

Proposition 2.10. *Let G_1 and G_2 be two bundle-like metrics with basic mean curvature forms k_1 and k_2 . Consider the canonical (up to a multiplicative positive constant) isomorphism:*

$$\zeta : H_{-k_2}^{0,2}(\mathcal{M}/\mathcal{F}) \rightarrow H_{-k_1}^{0,2}(\mathcal{M}/\mathcal{F})$$

given by $\zeta([\varphi^{0,2}]) = [e^f \varphi^{0,2}]$, where $\bar{\partial}_b f = k_1^{0,1} - k_2^{0,1}$. Then, $\zeta([\mathcal{E}_2^{0,2}(\mathcal{F})])$ and $[\mathcal{E}_1^{0,2}(\mathcal{F})]$ are proportional. In particular, the vanishing of the basic Euler $(0, 2)$ class does not depend on G , but just on \mathcal{F} .

2.4. Hodge decomposition. Let us continue to consider $(\mathcal{M}, \mathcal{F}, g)$ be a transversally Kählerian flow endowed with a bundle-like metric G and with the mean curvature k as in the previous subsection. Recently, in [10, 26], is studied a basic cohomology with respect to the modified differential of Lichnerowicz type $d_{b,k} = d_b - \frac{1}{2}k\wedge$, and a Hodge decomposition for basic forms and a Poincaré duality with respect to this cohomology for Riemannian foliations are obtained. Here, we give some similar results for transversally Kählerian flows with respect to the Dolbeault operator of Lichnerowicz type $\bar{\partial}_{b,k} = \bar{\partial}_b - k^{0,1}\wedge$.

According to [23], if $\varphi \in \Omega^r(\mathcal{M}/\mathcal{F})$ then $(\varphi\wedge)^* = (-1)^{n(r+1)} * (\varphi\wedge)^*$, where $(\varphi\wedge)^*$ denotes the adjoint of $\varphi\wedge$ as an operator on basic r -forms. Here $*$ is the basic star operator and the adjointness is with respect to the inner product in the space of basic forms, see [5, 6]. Taking into account that $(\mathcal{M}, \mathcal{F}, g)$ is transversally Kählerian and $k = k^{1,0} + k^{0,1}$, by equating the terms of the same components one gets

$$(2.18) \quad (k^{1,0}\wedge)^* = *(k^{0,1}\wedge)^*, \quad (k^{0,1}\wedge)^* = *(k^{1,0}\wedge)^*.$$

Using (2.18), by similar calculations as in the proof of Proposition 2.8. from [10] we obtain

Proposition 2.11. *We have the following identities for operators acting on the space $\Omega^{p,q}(\mathcal{M}/\mathcal{F})$:*

- (i) $(k^{1,0} \wedge)^* * = (-1)^{p+q} * (k^{0,1} \wedge)$, $(k^{0,1} \wedge)^* * = (-1)^{p+q} * (k^{1,0} \wedge)$;
- (ii) $*(k^{1,0} \wedge)^* = (-1)^{p+q+1} (k^{0,1} \wedge)^*$, $*(k^{0,1} \wedge)^* = (-1)^{p+q+1} (k^{1,0} \wedge)^*$;
- (iii) $\partial_{b,k}^* * = (-1)^{p+q+1} * \bar{\partial}_{b,k}$, $\bar{\partial}_{b,k}^* * = (-1)^{p+q+1} * \partial_{b,k}$;
- (iv) $* \partial_{b,k}^* = (-1)^{p+q} \bar{\partial}_{b,k}^*$, $* \bar{\partial}_{b,k}^* = (-1)^{p+q} \partial_{b,k}^*$.

Let us denote by $\Delta'_{b,k} = \partial_{b,k} \partial_{b,k}^* + \partial_{b,k}^* \partial_{b,k}$ and $\Delta''_{b,k} = \bar{\partial}_{b,k} \bar{\partial}_{b,k}^* + \bar{\partial}_{b,k}^* \bar{\partial}_{b,k}$ the basic complex Lichnerowicz Laplacians associated to k . Then similar arguments as in Proposition 2.3. and Theorem 3.1. from [10] leads to

Theorem 2.2. (*Hodge decomposition for Dolbeault $\bar{\partial}_{b,k}$ -cohomology.*) *We have the following orthogonal direct decomposition*

$$(2.19) \quad \Omega^{p,q}(\mathcal{M}/\mathcal{F}) = \ker \Delta''_{b,k} \oplus \text{im } \bar{\partial}_{b,k} \oplus \text{im } \bar{\partial}_{b,k}^*.$$

2.5. Basic Lichnerowicz cohomology of transversally LCK foliations. In this subsection we consider a version of locally conformally Kähler manifold notion, see [31, 32, 33], for transversally Kählerian foliations, [3], and we investigate some problems concerning to basic Lichnerowicz cohomology of locally conformally transversally Kählerian foliations.

Definition 2.2. ([3]). A locally conformally transversally Kählerian foliation, briefly transversally LCK foliation, is an transversally Hermitian foliation $(\mathcal{M}, \mathcal{F}, g)$ for which an open covering $\{U_i\}$ exists, and for each i a basic function $\sigma_i : U_i \rightarrow \mathbb{R}$ such that $\tilde{g} = e^{-\sigma_i}(g|_{U_i})$ is a transverse Kähler metric on U_i , called a *locally conformally transverse Kähler metric*.

Similar to Proposition 1.1. from [31], the foliation \mathcal{F} is a transversally LCK foliation iff its transverse bundle $Q\mathcal{F}$ has a Kähler metric which is locally conformally with a foliated Hermitian metric. It is easy to see that $\theta|_{U_i} = d_b \sigma_i$ defines a global d_b -closed 1-form, and that $(\mathcal{M}, \mathcal{F}, \omega)$ has the characteristic property

$$(2.20) \quad d_b \omega = \theta \wedge \omega,$$

where ω is the basic Hermitian form on $(\mathcal{M}, \mathcal{F})$. If we take $U_i = \mathcal{M}$, then $(\mathcal{M}, \mathcal{F}, \omega)$ is called globally conformally Kählerian foliation. The basic form θ is called the basic Lee form of $(\mathcal{M}, \mathcal{F}, \omega)$. It is exact iff $(\mathcal{M}, \mathcal{F}, \omega)$ is globally conformal Kählerian foliation.

Now, if $(\mathcal{M}, \mathcal{F}, \omega)$ is a transversally LCK foliation with θ its basic Lee form, then due to (2.20) we have $d_{b,\theta} \omega = 0$. Therefore, ω represents a cohomology class in the basic Lichnerowicz complex $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta})$.

Definition 2.3. The basic Lichnerowicz cohomology class $[\omega]_L$ of ω is called the basic Lichnerowicz class of the transversally LCK foliation $(\mathcal{M}, \mathcal{F}, \omega)$.

This invariant is a basic version of Morse-Novikov class of LCK manifolds, see [22].

Definition 2.4. If $(\mathcal{M}, \mathcal{F}, \omega)$ is a transversally LCK foliation then the cohomology class $[\omega]_{BCL} \in H_{BCL}^{1,1}(\mathcal{M}/\mathcal{F})$ is called the *basic Bott-Chern-Lichnerowicz class* of $(\mathcal{M}, \mathcal{F}, \omega)$.

Thus, for any transversally LCK foliation we have three basic cohomological invariants:

- the basic Lee class $[\theta] \in H^1(\mathcal{M}/\mathcal{F})$;
- the basic Lichnerowicz class $[\omega]_L \in H_\theta^2(\mathcal{M}/\mathcal{F})$;
- the basic Bott-Chern-Lichnerowicz class $[\omega]_{BCL} \in H_{BCL}^{1,1}(\mathcal{M}/\mathcal{F})$.

Now, using an argument inspired from [15], we briefly present another basic cohomology associated to transversally LCK foliations which is connected with the basic Lichnerowicz cohomology of transversally LCK foliations. Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with θ its basic Lee form. We consider the basic closed 1-forms θ_0 and θ_1 defined by

$$(2.21) \quad \theta_0 = m\theta \text{ and } \theta_1 = (m+1)\theta, \quad m \in \mathbb{R}.$$

Denote by $H_{\theta_0}^\bullet(\mathcal{M}/\mathcal{F})$ and $H_{\theta_1}^\bullet(\mathcal{M}/\mathcal{F})$ the basic Lichnerowicz cohomologies of the basic complexes $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{\theta_0})$ and $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{\theta_1})$, respectively.

Now, let $\widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) = \Omega^k(\mathcal{M}/\mathcal{F}) \oplus \Omega^{k-1}(\mathcal{M}/\mathcal{F})$ and $\widehat{d}_b : \widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) \rightarrow \widehat{\Omega}^{k+1}(\mathcal{M}/\mathcal{F})$ be the basic differential operator defined by

$$(2.22) \quad \widehat{d}_b(\varphi, \psi) = (d_{b, \theta_1} \varphi - \omega \wedge \psi, -d_{b, \theta_0} \psi).$$

Using (2.20), by direct calculus it follows $\widehat{d}_b^2 = 0$. Thus, we can consider the basic complex $(\widehat{\Omega}^\bullet(\mathcal{M}/\mathcal{F}), \widehat{d}_b)$ and the associated basic cohomology $\widehat{H}^\bullet(\mathcal{M}/\mathcal{F})$. We have the following result which relates $\widehat{H}^\bullet(\mathcal{M}/\mathcal{F})$ with basic Lichnerowicz cohomologies $H_{\theta_0}^\bullet(\mathcal{M}/\mathcal{F})$ and $H_{\theta_1}^\bullet(\mathcal{M}/\mathcal{F})$.

Proposition 2.12. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with θ its basic Lee form. Suppose that $i^k : \Omega^k(\mathcal{M}/\mathcal{F}) \rightarrow \widehat{\Omega}^k(\mathcal{M}/\mathcal{F})$ and $\pi_2^k : \widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) \rightarrow \Omega^{k-1}(\mathcal{M}/\mathcal{F})$ are homomorphisms of $\mathcal{F}_b(\mathcal{M}, \mathbb{R})$ -modules defined by*

$$i^k(\varphi) = (\varphi, 0) \text{ and } \pi_2^k(\varphi, \psi) = \psi,$$

for $\varphi \in \Omega^k(\mathcal{M}/\mathcal{F})$ and $\psi \in \Omega^{k-1}(\mathcal{M}/\mathcal{F})$. Then:

i) *The mappings i^k and π_2^k induce an exact sequence of basic complexes*

$$0 \longrightarrow (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b, \theta_1}) \xrightarrow{i^k} (\widehat{\Omega}^\bullet(\mathcal{M}/\mathcal{F}), \widehat{d}_b) \xrightarrow{\pi_2^k} (\Omega^{\bullet-1}(\mathcal{M}/\mathcal{F}), -d_{b, \theta_0}) \longrightarrow 0.$$

ii) *This exact sequence induces a long exact basic cohomology sequence*

$$\dots \longrightarrow H_{\theta_1}^k(\mathcal{M}/\mathcal{F}) \xrightarrow{(i^k)^*} \widehat{H}^k(\mathcal{M}/\mathcal{F}) \xrightarrow{(\pi_2^k)^*} H_{\theta_0}^{k-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{-\delta^{k-1}} H_{\theta_1}^{k+1}(\mathcal{M}/\mathcal{F}) \longrightarrow \dots,$$

where the connector homomorphism $-\delta^{k-1}$ is defined by

$$(2.23) \quad (-\delta^{k-1})[\varphi] = [\varphi \wedge \omega], \quad \forall [\varphi] \in H_{\theta_0}^{k-1}(\mathcal{M}/\mathcal{F}).$$

From the above proposition, we obtain

Corollary 2.3. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with θ its basic Lee form and such that the basic Lichnerowicz cohomology groups $H_{\theta_0}^k(\mathcal{M}/\mathcal{F})$ and $H_{\theta_1}^k(\mathcal{M}/\mathcal{F})$ have finite dimension, for all k . Then, the basic cohomology group $\widehat{H}^k(\mathcal{M}/\mathcal{F})$ has also finite dimension, for all k , and*

$$(2.24) \quad \widehat{H}^k(\mathcal{M}/\mathcal{F}) \cong \frac{H_{\theta_1}^k(\mathcal{M}/\mathcal{F})}{\text{Im } \delta^{k-2}} \oplus \ker \delta^{k-1},$$

where $\delta^k : H_{\theta_0}^k(\mathcal{M}/\mathcal{F}) \rightarrow H_{\theta_1}^{k+2}(\mathcal{M}/\mathcal{F})$ is the homomorphism given by (2.23).

Corollary 2.4. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with θ its basic Lee form such that the dimensions of the basic cohomology groups $H_{\theta_0}^k(\mathcal{M}/\mathcal{F})$ and $H_{\theta_1}^k(\mathcal{M}/\mathcal{F})$ are*

finite, for all k . Suppose that ω is $d_{b,\theta}$ -exact, that is, there exists a basic 1-form ω' on $(\mathcal{M}, \mathcal{F})$ satisfying $\omega = d_b \omega' - \theta \wedge \omega'$. Then, for all k , we have

$$(2.25) \quad \widehat{H}^k(\mathcal{M}/\mathcal{F}) \cong H_{\theta_1}^k(\mathcal{M}/\mathcal{F}) \oplus H_{\theta_0}^{k-1}(\mathcal{M}/\mathcal{F}).$$

Let us consider further $(\mathcal{M}, \mathcal{F}, \omega)$ to be a transversally Kählerian foliation. Then the basic Kähler form ω determines the basic Kähler class $[\omega] \in H^{1,1}(\mathcal{M}/\mathcal{F})$, and the difference of basic Kähler forms which have the same basic Kähler class is measured by a basic potential f :

$$\omega_1 - \omega = \partial_b \bar{\partial}_b f$$

see Proposition 3.5.1. from [5]. (This also follows from $\partial_b \bar{\partial}_b$ -Lemma, [21]). Thus the space of all basic Kähler structures on a transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ is locally modeled on $H^{1,1}(\mathcal{M}/\mathcal{F}, \mathbb{R}) \times (\mathcal{F}_b(\mathcal{M})/\text{const})$. A similar local description exists for the set of all LCK-structures on a given transversally holomorphic foliation, if we fix the basic cohomology class $[\theta]$ of a basic Lee form. Using the basic Bott-Chern-Lichnerowicz class $[\omega]_{BCL} \in H_{BCL}^{1,1}(\mathcal{M}/\mathcal{F})$ of an LCK-basic form ω , similarly to [22], we can obtain that the difference of two LCK-basic forms in the same basic Bott-Chern-Lichnerowicz class is expressed by a basic potential, just like in transversally Kähler case, and the set of all LCK-structures on a given transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ is locally parametrized by

$$(2.26) \quad H_{BCL}^{1,1}(\mathcal{M}/\mathcal{F}) \times (\mathcal{F}_b(\mathcal{M})/\text{Ker } d_{b,\theta} d_{b,\theta}^c),$$

where $d_{b,\theta} d_{b,\theta}^c = -2\sqrt{-1} \partial_b \bar{\partial}_b$.

In order to find a connection between basic Bott-Chern cohomology, basic Lichnerowicz cohomology and basic holomorphic cohomology of a flat foliated complex line bundle E associated with θ , we briefly recall some definitions concerning to foliated bundles and basic connections, see [6, 14, 18].

Let $G \hookrightarrow P \rightarrow M$ be a principal bundle with structural group $G \subset \text{GL}(n, \mathbb{C})$. The group G acts on P on the right and on its Lie algebra \mathcal{G} by the adjoint representation Ad i.e., for $g \in G$ and $X \in \mathcal{G}$, $\text{Ad}_g(X) = gXg^{-1}$. We say that a principal G -bundle $P \rightarrow (\mathcal{M}, \mathcal{F})$ is a *foliated principal bundle* if it is equipped with a foliation \mathcal{F}_P (the *lifted foliation*) such that the distribution $T\mathcal{F}_P$ is invariant under the right action of G , is transversal to the tangent space to the fiber, and projects to $T\mathcal{F}$. A connection ω on P is called *adapted* to \mathcal{F}_P if the associated horizontal distribution contains $T\mathcal{F}_P$. An adapted connection γ is called a *basic connection* if it is basic as a \mathcal{G} -valued form on (P, \mathcal{F}_P) .

Let $E \rightarrow (\mathcal{M}, \mathcal{F})$ be a complex vector bundle defined by a cocycle $\{U_i, g_{ij}, G\}$ where $\{U_i\}$ is an open cover of \mathcal{M} and $g_{ij} : U_i \cap U_j \rightarrow G \subset \text{GL}(n, \mathbb{C})$ are the transition functions. To such a vector bundle we can always associate a principal G -bundle $P \rightarrow (\mathcal{M}, \mathcal{F})$ whose fibre is the group G and the transition functions are exactly the g_{ij} (viewed as translations on G). The complex vector bundle $E \rightarrow (\mathcal{M}, \mathcal{F})$ is *foliated* if E is associated to a foliated principal G -bundle $P \rightarrow (\mathcal{M}, \mathcal{F})$. Let $\Omega^\bullet(\mathcal{M}, E)$ denote the space of forms on $(\mathcal{M}, \mathcal{F})$ with coefficients in E . If a connection form γ on P is adapted, then we say that an associated covariant derivative operator ∇ on $\Omega^\bullet(\mathcal{M}, E)$ is *adapted* to the foliated bundle. We say that ∇ is a *basic connection* on E if in addition the associated curvature operator ∇^2 satisfies $i_X \nabla^2 = 0$ for every $X \in \Gamma(\mathcal{F})$. Note that ∇ is basic if γ is basic. Let $\Gamma(E)$ denote the smooth sections of E , and let ∇ denote a basic connection on E . We say that a section $s : \mathcal{M} \rightarrow E$ is a *basic section* if and only if $\nabla_X s = 0$ for all $X \in \Gamma(\mathcal{F})$. Let ∇_b denote the basic connection and $\Gamma_b(E)$ denote the space of basic sections of E .

Now, let us consider E to be a foliated complex line bundle over the transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ with a flat basic connection ∇_b . We denote by $\Omega^{p,q}(\mathcal{M}/\mathcal{F}, E)$ the set of all basic (p, q) -forms on $(\mathcal{M}, \mathcal{F})$ with coefficients in E . Consider the basic complex

(2.27)

$$\Omega^{p-1, q-1}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\partial_{b,E} \bar{\partial}_{b,E}} \Omega^{p,q}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\partial_{b,E} \oplus \bar{\partial}_{b,E}} \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}, E) \oplus \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}, E),$$

where $\partial_{b,E}$ and $\bar{\partial}_{b,E}$ denote the $(1, 0)$ and $(0, 1)$ -parts of the basic connection operator $\nabla_b : \Omega^\bullet(\mathcal{M}/\mathcal{F}, E) \rightarrow \Omega^{\bullet+1}(\mathcal{M}/\mathcal{F}, E)$. The cohomology of (2.27) denoted by $H_{BC}^{p,q}(\mathcal{M}/\mathcal{F}, E)$ is called *the basic Bott-Chern cohomology of $(\mathcal{M}, \mathcal{F})$ with coefficients in E* .

Definition 2.5. Let $(\mathcal{M}, \mathcal{F}, \omega, \theta)$ be a transversally LCK foliation, and E its foliated weight bundle, that is, a trivial complex foliated line bundle with the flat basic connection $d_b - \theta$. Consider ω as a closed E -basic $(1, 1)$ -form on $(\mathcal{M}, \mathcal{F})$. Its basic Bott-Chern class $[\omega]_{BC} \in H_{BC}^{1,1}(\mathcal{M}/\mathcal{F}, E)$ is called *the basic Bott-Chern class of the transversally LCK foliation $(\mathcal{M}, \mathcal{F}, \omega, \theta)$* .

Now, similarly to [22], we give a characterization of $H_{BC}^{1,1}(\mathcal{M}/\mathcal{F}, E)$ in terms of basic Lichnerowicz cohomology of $(\mathcal{M}, \mathcal{F}, \theta)$ and holomorphic basic cohomology of the foliated weight bundle E .

The holomorphic basic cohomology of a foliated bundle E can be realized as cohomology of the complex

$$(2.28) \quad \Gamma_b(E) = \Omega^{0,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\bar{\partial}_{b,E}} \Omega^{0,1}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\bar{\partial}_{b,E}} \Omega^{0,2}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\bar{\partial}_{b,E}} \dots$$

If E is equipped with a flat basic connection, then $\partial_{b,E} : \Omega^{0,1}(\mathcal{M}/\mathcal{F}, E) \rightarrow \Omega^{1,1}(\mathcal{M}/\mathcal{F}, E)$ induces a map

$$(2.29) \quad H^1(\mathcal{M}/\mathcal{F}, \mathcal{E}) \xrightarrow{\partial_{b,E}^*} H_{BC}^{1,1}(\mathcal{M}/\mathcal{F}, E)$$

from the holomorphic basic cohomology of the underlying holomorphic foliated bundle (denoted as \mathcal{E}) to the basic Bott-Chern cohomology. The basic complex

$$(2.30) \quad \Gamma_b(E) = \Omega^{0,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\nabla_b^{1,0}} \Omega^{1,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\nabla_b^{1,0}} \Omega^{2,0}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\nabla_b^{1,0}} \dots$$

computes the holomorphic basic cohomology of a foliated bundle \mathcal{E}' with a holomorphic structure defined by the complex conjugate of the $\nabla_b^{1,0}$ -part of the basic connection. When the bundle E is real, we have $\mathcal{E} \approx \mathcal{E}'$. Then the cohomology of the basic complex (2.30) is naturally identified with $\overline{H^\bullet(\mathcal{M}/\mathcal{F}, \mathcal{E})}$. The map $\bar{\partial}_{b,E} : \Omega^{1,0}(\mathcal{M}/\mathcal{F}, E) \rightarrow \Omega^{1,1}(\mathcal{M}/\mathcal{F}, E)$ defines a homomorphism

$$(2.31) \quad \overline{H^1(\mathcal{M}/\mathcal{F}, \mathcal{E})} \xrightarrow{\bar{\partial}_{b,E}^*} H_{BC}^{1,1}(\mathcal{M}/\mathcal{F}, E)$$

which is entirely similar to (2.29).

Following step by step the proof of Theorem 4.7. from [22], we obtain a basic analogue result, which allows to compute the basic Bott-Chern cohomology classes in terms of holomorphic basic cohomology and basic Lichnerowicz cohomology.

Theorem 2.3. *Let $(\mathcal{M}, \mathcal{F})$ be a transversally holomorphic foliation and $E_{\mathbb{R}}$ a trivial real foliated line bundle with flat basic connection $d_b - \theta$, where θ is a real closed basic 1-form.*

Denote by E its complexification, and let \mathcal{E} be the underlying holomorphic bundle. Then there is an exact sequence

$$(2.32) \quad H^1(\mathcal{M}/\mathcal{F}, \mathcal{E}) \oplus \overline{H^1(\mathcal{M}/\mathcal{F}, \mathcal{E})} \xrightarrow{\partial_{b,E}^* + \overline{\partial}_{b,E}^*} H_{BC}^{1,1}(\mathcal{M}/\mathcal{F}, E) \xrightarrow{\nu} H_\theta^2(\mathcal{M}/\mathcal{F}),$$

where $H_\theta^2(\mathcal{M}/\mathcal{F})$ is the basic Lichnerowicz cohomology, ν a tautological map, and the first arrow is obtained as a direct sum of (2.29) and (2.31).

From the above theorem, we immediately obtain

Corollary 2.5. *Let $(\mathcal{M}, \mathcal{F}, \omega, \theta)$ be a transversally LCK foliation, E the corresponding flat foliated bundle, and \mathcal{E} the underlying holomorphic bundle. Assume that $H^1(\mathcal{M}/\mathcal{F}, \mathcal{E}) = 0$ and $H_\theta^2(\mathcal{M}/\mathcal{F}) = 0$. Then $H_{BC}^{1,1}(\mathcal{M}/\mathcal{F}) = 0$.*

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